THRESHOLD OF PARAMETRIC EXCITATION OF WAVES ON A FLUID SURFACE

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One of the interesting effects generated during the action of a variable external field of large amplitude (pumping) on matter is parametric instability. At the present time this instability has been observed in ferrites, antiferromagnets, dielectrics, plasmas, and other media. Despite the variety of phenomena, in all cases parametric instability has very much in common — decaying instability of waves in the medium, satisfying the conservation laws:

$$\mathbf{k}_p = \mathbf{k}_1 + \mathbf{k}_2, \quad \omega_p = \omega_{k_1} + \omega_{k_2}.$$

Here $\mathbf{k}_p$ and ω_p are the wave vector and frequency of the pumping field, while $\mathbf{k}_1, \mathbf{k}_2$ and ω_{k_1}, ω_{k_2} are the wave vectors and frequencies of the generated waves. In the absence of pumping the wave amplitudes in the medium decay as $\exp(-\gamma_k t)$. Pumping leads to creation of waves, while the frequency of their creation is proportional to the amplitude of the pumping field. Thus, for an excess of the pumping field over the threshold value wave creation predominates over their dissipation. As a result the amplitudes of parametric waves (PW) increase exponentially, and their growth can be restricted only by nonlinear wave interaction.

The present study is devoted to studying the threshold of parametric wave excitation on the surface of water, excited by the pressure of an ac electric field $E^2/8\pi$. The basic experimental device is a transparent vessel filled with water, of sizes 90 × 90 × 10 cm, over which is found a metallic electrode. A voltage up to 13 kV and a frequency up to 14 GHz can be applied between the electrode and the water, and the gap was around 1 cm. A laser beam was propagated vertically through the water layer, so that the slope of the water surface could be determined from its inclination.

Below we explain the connection between the wave damping γ_k and the pumping amplitude, and the physical generation mechanisms of this damping are discussed. The answer is almost trivial in an infinite system: $\gamma_k = |V_k|$ (V_k is the interaction coefficient of waves with the pumping). The system can be considered infinite if its size L exceeds substantially the wave mean free path v_k/γ_k (v_k is the group velocity). Under the conditions of our experiment $L \sim v_k/\gamma_k$, and therefore the threshold could depend not only on the damping of free waves γ_k, but also on the friction with the wall and on the fact that the wave spectrum in the vessel is discrete. We show that these factors are not essential, and the instability threshold in the device is near the threshold in an infinite system.

The damping of surface waves in an ideal fluid is $\gamma_k = 2\nu k^2$ (ν is the kinematic viscosity) [1]. The measured damping was larger by several times. This is explained by the fact that a film is formed on the water surface of undissolved grains adsorbed from air, and, possibly, of undissolved material contained in the volume of the fluid. The presence of the film leads to a variation in the surface tension coefficient and large additional wave dissipation [2]. The occurrence of a surface film and its strong effect on wave damping in water is a quite general property, noted in studies of wave excitation by wind on the ocean surface [3].

The experimentally measured quantities were damping, thresholds of wave excitation, and the surface tension coefficient. This made it possible to establish the values of the phenomenological constants characterizing the elasticity of the adsorbed film.

1. Parametric Instability of a Finite System. Parametric wave excitation is conveniently described within the classical Hamilton formalism [4, 5]. In this case the equations of motion for complex wave amplitudes $a(\mathbf{r})$ are $\partial a(\mathbf{r})/\partial t = -i\delta H/(\delta a^*(\mathbf{r}))$. The Hamiltonian $H\{a, a^*\}$ for waves on a fluid surface and the relation of normal canonical amplitudes a, a^* with the amplitude of surface oscillations and the velocity potential were determined in [6]. We assume that the wave amplitude is small, and restrict ourselves to second-order terms

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in a, a^* in the Hamiltonian expansion. In the absence of an electric field (pumping) there may exist weakly damped waves in the medium. In an infinite system these are plane waves $a(\mathbf{r}) \exp(i\mathbf{kr})$ with frequency

$$\omega_k = \sqrt{gk + \alpha k^3/\rho} \tag{1.1}$$

(g is the free-fall acceleration, α is the surface tension coefficient, and ρ is the density). In a confined system, such as a rectangular vessel, the normal eigenfunctions are

$$\varphi_{pq} = \frac{2}{L} \cos(px) \cos(qy), \quad p = \frac{\pi n}{L}, \quad q = \frac{\pi m}{L} \tag{1.2}$$

(L is the size of the square, and n, m are integers), and the eigenfrequencies are given by Eq. (1.1) with $k = \sqrt{p^2 + q^2}$ [1]. We expand $a(\mathbf{r})$ in the eigenfunctions $\varphi_\lambda(\mathbf{r})$. In the general case the equations of motion for the expansion coefficients in the presence of parametric pumping are within the linear approximation

$$\frac{\partial a_\lambda}{\partial t} + (\gamma_\lambda + i\omega_\lambda) a_\lambda + i \sum_{\lambda'} V_{\lambda\lambda'} e^{-2i\omega_0 t} a_{\lambda'}^* = f_\lambda e^{-i\omega_0 t}.$$

Here we introduced wave damping and a coupling coefficient with pumping $V_{\lambda\lambda'}$; $\omega_0 = \omega_p/2$ is the electric field frequency, and f_λ is a generalized force, acting with frequency ω_0. It is caused by inhomogeneities of the electric field $E(\mathbf{r}, t)$, and is small under experimental conditions. In an infinite homogeneous medium with a homogeneous pumping field $\lambda = \mathbf{k}$, $V_{\lambda\lambda'} = V_k \delta(\mathbf{k} + \mathbf{k}')$, and in a square vessel with homogeneous pumping $\lambda = (p, q)$, $V_{\lambda\lambda'} = V_k \delta_{pp'} \delta_{qq'}$. The matrix element V_k was calculated in [6, 7]:

$$V_k = \frac{E^2 k^2}{32\pi\rho\omega_k} \operatorname{cth}(kh) \tag{1.3}$$

(h is the value of the gap between the electrode and the water surface. In a homogeneous electric field $f_\lambda = 0$, and therefore the amplitudes a_λ, a_λ^* increase exponentially with an increment ν_λ:

$$a_\lambda, \ a_{\bar\lambda}^* \sim e^{\nu_\lambda t}, \ \nu_\lambda = -\gamma_\lambda + \sqrt{|V_\lambda|^2 - \bar\omega_\lambda^2}, \ \bar\omega_\lambda = \omega_\lambda - \omega_0, \ \bar\lambda = (-p, -q). \tag{1.4}$$

It is understood that growth is possible only for quite high pumping amplitude, when

$$|V_\lambda|^2 \geqslant \gamma_\lambda + \bar\omega_\lambda^2. \tag{1.5}$$

The instability evolves when condition (1.5) is satisfied for one type of oscillations a_λ. In an infinite system the spectrum of oscillations a_λ is continuous, and the instability threshold is determined from the condition

$$|V_k| = \gamma_\lambda \ \text{for} \ \omega_k = \omega_0. \tag{1.6}$$

In an unbounded fluid γ_k and ω_k are isotropic, therefore the threshold is achieved directly for all waves with wave vectors near resonance $\omega_k = \omega_0$. For an electric field exceeding the threshold value E_c the linear theory of parametric wave excitation considered here is not valid.

It follows from Eqs. (1.4) that for electric field values less than the threshold the wave amplitude a_λ tends to zero. Account of the small induced force f_λ leads to a finite stationary amplitude a_λ up to the threshold:

$$a_\lambda = \frac{(\gamma_{\bar\lambda} - i\bar\omega_{\bar\lambda}) f_\lambda + iV_{\bar\lambda}^* f_{\bar\lambda}^*}{(\gamma_\lambda + i\omega_\lambda)(\gamma_{\bar\lambda} - i\bar\omega_{\bar\lambda}) - V_{\bar\lambda}^* V_\lambda} e^{-i\omega_0 t}, \tag{1.7}$$

while the process occurs exponentially with a decrement in the resonanance area

$$|\nu| = \frac{k^2 \operatorname{cth}(kh)}{32\pi\rho\omega_k} (E_c^2 - E^2). \tag{1.8}$$

In a finite system the parameter λ runs through a discrete series of values, and the spectrum ω_λ is also discrete. For example, for waves in a square vessel the admitted λ values can be located only in sites of a square lattice with a period π/L. In this case, generally speaking, the condition $\omega_\lambda = 0$ cannot be satisfied, and the threshold in a finite system is higher than in an infinite system. If the size of the system L is large in comparison with the wave length, then V_λ and γ_λ depend weakly on λ near the resonance surface $\omega_\lambda = \omega_0$, and therefore the threshold will be achieved first for those λ for which the frequency separation $|\bar\omega_\lambda|$ is minimal. We estimate $\min|\bar\omega_\lambda|$. For this it is convenient to transform in λ-space to polar coordinates and determine at which distance from the resonance neighborhood is found the closest point of λ-space. The separation Δλ from the resonance neighborhood to the nearest point of λ-space can be obtained from the equality condition of the areas

$$\frac{1}{4}2\pi k\Delta\lambda=\left(\frac{\pi}{L}\right)^2, \tag{1.9}$$

whence $\Delta\lambda=\frac{2\pi}{L}\frac{1}{kL}$. The factor 1/4 in (1.9) arises from the fact that one must consider eigenfunctions (1.2) with p > 0, q > 0. Knowing Δλ, it is easily found that $\min|\bar{\omega}_\lambda|\leqslant\frac{\partial\omega_k}{\partial k}\Delta\lambda=\frac{v_k}{kL}\frac{2\pi}{L}$ (v_k is the group velocity). The parametric instability threshold is reached for

$$\gamma_\lambda^2\leqslant|V_\lambda|^2\leqslant\gamma_\lambda^2+\left(\frac{v_k}{kL}\frac{2\pi}{L}\right)^2=\gamma_\lambda^2\left[1+\left(\frac{l}{L}\frac{2\pi}{kL}\right)^2\right].$$

Here $\ell = v_k/\gamma_k$ is the mean free path of waves. Thus, even when ℓ is comparable to or exceeds the system size, the threshold amplitude of the field is near the value for an infinite system. The threshold is substantially enhanced in comparison with the threshold for an infinite system only for very small system sizes $L \lesssim \sqrt{\ell d}$ ($d = 2\pi/k$ is the wavlength). The opposite inequality $L \gg (\ell d)^{1/2}$ is satisfied in our experiments, therefore the effect of spectrum discreteness on the threshold can be neglected.

Wave dissipation at the boundary also leads to further contribution to damping [1]: $\Delta\gamma_S=\frac{3}{2\sqrt{2}}\frac{\sqrt{\nu\omega}}{L}$. This contribution is isotropic, and under experimental conditions is roughly 10^{-2} of the total damping γ_k.

It seemed initially that damping of surface waves is due to bulk viscous friction [1]

$$\gamma_k = 2\nu k^2. \tag{1.10}$$

Experiment showed, however, that the threshold of parametric instability is substantially higher than that calculated by Eqs. (1.6), (1.10), and has an unknown frequency dependence. The enhanced wave damping is related to the known properties of water adsorbed on the surface of a nondissolving material. The presence of a film of adsorbed particles leads to formation of a viscous layer near the fluid surface due to friction with the film. In the limiting case of a dense incompressible film the wave damping equals [1, 2]

$$\gamma_k = \sqrt{\omega_k\nu k^2/8}. \tag{1.11}$$

Fluid motions lead to inhomogeneities in the distribution of adsorbed particles over the surface, generating an additional restoring force acting on the surface. This force is proportional to the gradient of the surface tension coefficient α, which, in turn, depends on the adsorbed particle concentration c. Assuming that the wave quality is high ($\gamma_k \ll \omega_k$), one can obtain an expression for wave damping on the surface of an unbounded fluid [2]

$$\frac{\gamma_k}{\omega_k}=\mathrm{Re}\left[\frac{2\nu k^2}{\omega_k}+\frac{i\beta k^3}{2\rho\omega_k^2}\right]\left[1-(1-i)\sqrt{\frac{\omega_k}{2\nu k^2}\frac{\beta k^3}{\rho\omega_k^2}}\right]^{-1}, \tag{1.12}$$

where $\beta = \partial\alpha/\partial \ln c$ is a parameter characterizing the elasticity of the film.

In the limiting case $(\nu k^2/\omega_k)^{3/4}\gg\beta k^3/\rho\omega_k^2$ the answer (1.10) follows from (1.12), i.e., the effect of the film is negligibly small. In the parameter region $\nu k^2/\omega_k\ll\beta k^3/\rho\omega_k^2\ll(\nu k^2/\omega_k)^{1/2}$ the damping is $\gamma_k=\frac{\omega_k}{2\sqrt{2}}\left(\frac{\beta k^3}{\rho\omega_k^2}\right)^2\sqrt{\frac{\omega_k}{\nu k^2}}$. Finally, in the limit of a rigid film $\beta k^3/\rho\omega_k k^2\gg(\nu k^2/\omega_k)^{1/2}$ the damping tends to (1.11): $\gamma_k=\frac{1}{2\sqrt{2}}\sqrt{\omega_k\nu k^2}\left[1+\frac{\sqrt{2\nu k^2/\omega_k}}{\beta k^3/\rho\omega_k^2}\right]$.

The maximum damping for a fixed frequency is reached for a value of the quantity β, depending on the concentration c_0, when $\beta k^3/\rho\omega_k^2=\sqrt{2\nu k^2/\omega_k}$. The maximum damping value is twice as large as the limiting value.

2. Experimental Device. Method and Experimental Results. In the present study we have determined experimentally the threshold voltage of the electric field E_c of surface wave (SW) excitation for various pumping frequencies, as well as the dependence of wave damping γ_k on the wave vector. The presence of particles on the water surface leads to a decrease in the

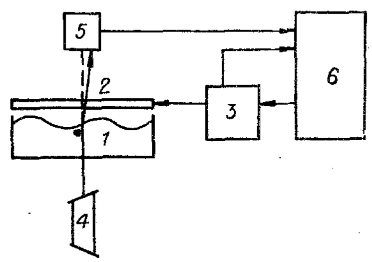

Fig. 1

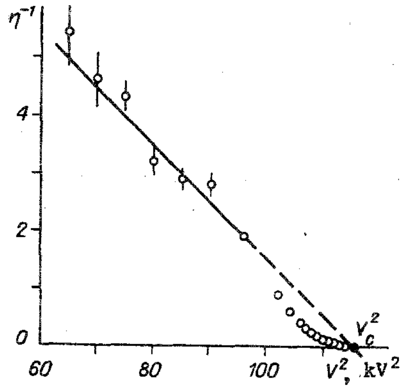

Fig. 2

coefficient α [8]. To determine the wave dispersion law (1.1) we measured the quantity α.

The device is illustrated schematically in Fig. 1. The fluid is in vessel 1 (90 × 90 × 9 cm). A planar metallic electrode 2 (90 × 90 cm) was placed over the water surface, the gap was roughly 1 cm, and the electrode was horizontal with accuracy of 200 μm. An ac electric voltage with amplitude near 10 kV and frequency near 10 Hz was applied between the water and the electrode. The electric field created a high-voltage source 3, being an intense two-channel amplifier with amplitude modulation. The source had the following parameters: voltage amplitude from 0 to 13 kV in the frequency region from 0 to 14 Hz, amplitude nonlinearity less than 0.1%/10 kV, amplitude stability 0.3%, coefficient of harmonics <0.1%, modulation frequency 2 kHz, and harmonic modulation amplitude <1%. This source made it possible to create an electric field voltage up to 10 kV/cm, capable of exciting surface waves in the absence of an electric probe between the water and the electrode. The electric field was recorded by measuring the voltage applied directly between the electrode and the water in terms of the voltage duration. The surface oscillations were fixed by an optical laser device, consisting of a laser 4 and a differential photodevice 5.

The automation system 6 was regulated by the electric voltage amplitude, and during the experiment allowed the selection and handling of data, entering the laser device and the high voltage source [9]. The experiment included the selection and visual control of data on the measurement time, harmonic fluid oscillations, and high voltage. In this case was realized the operative data processing and their maintenance for subsequent statistical treatment.

The measurement errors are basically statistical in nature: the spread of high voltage harmonics was determined by the discreteness of the analog–digital transformation, the noisy fluid oscillations were generated by structural vibrations, etc. The effect of excited waves on the high-voltage source was negligibly small (<0.1%) up to the probe.

The statistical treatment of experimental data, such as the time dependence of the wave amplitude, the voltage dependence of the standard SW amplitude, the frequency dependence of the electric field intensity, etc., was carried out by tracing regression curves through the experimental points. The curve parameters and their errors were determined by the highest likelihood method, while taking into account the error of both coordinates of the experimental points [10]. For example, to determine V_c from the voltage dependence of the stationary SW amplitude we used a regression curve with parameter V_c.

To determine the threshold of parametric wave excitation we measured the SW amplitude η for various values of the amplitude V and the frequency ω of the electric voltage. Since the pressure $E^2(t)/8\pi$ varied with frequency 2ω, the parametric wave had frequency $\omega_p/2 = \omega$. Therefore η was assumed proportional to the value of the harmonic signal ω, characteristic of the laser device. The amplitude V was found from the value of the first harmonic of the electric voltage.

The measurement of the threshold field amplitude was carried out as follows. At fixed frequency ω an electric voltage with amplitude V around 8 kV, which is quite below the threshold, was suppressed. In this case the amplitudes of all oscillation harmonics of the fluid surface are comparable with the noise. The electric voltage amplitude was then enhanced with a variable step of 50-1000 V, depending on the closeness to the threshold, and the measurement was carried out of the amplitude of the first wave harmonic η as a function of time.

When the voltage amplitude approaches the threshold value of the amplitude of the first wave harmonic, the values of the remaining harmonics remained at the noise level (less than 0.1% of the value of the first harmonic near the threshold). For fixed electric voltage the wave amplitude tended exponentially to its stationary value. Each new step of varying the voltage was carried out subsequently, as the fluid oscillations are established. The establishment time $1/|\nu|$ varied from 10 sec to 40 min, depending on the nearness to the threshold. From the data were determined the stationary wave amplitude η_0, the decrement $|\nu|$, and the electric voltage amplitude V. From Eqs. (1.7), (1.8) of the linear theory we obtain that in approaching threshold the stationary wave amplitude η_0, found in resonance with the pumping, increases as the hyperbola $\eta_0 \sim |f|/|\nu| \sim 1/(V_c^2 - V^2)$, while the $1/\eta_0$ drops linearly (Fig. 2). From the intersection point V_c^2 of the straight line with the V^2 axis we determined the threshold V_c with accuracy 0.1-0.3%. In the direct neighborhood of the threshold (around 200 V) was observed a sharp deviation from the linear theory. The SW relaxed nonexponentially to the stationary value, and η_0 increased according to the law $\eta_0 \sim (V_c - V)^{-\delta}$, where δ = 2-2.5. Since the linear phases of instability were considered, the points mentioned were excluded.

The measurement accuracy of the threshold voltage E_c was determined by the measurement error of the gap h (around 10%). To decrease the error we used the following method. Varying the gap with a step of 3000 μm, several threshold value measurements were carried out for the fixed frequency f = 10 GHz. Assuming that SW excitation occurred for identical values of the electric field, we found h from the proportionality condition of V_c along the gap at all measurements. The error decreased to 1.5%.

Another method of measuring the threshold field amplitude consisted of finding E_c^2 from the dependence (1.8) of the linear theory: $E_c^2 = E^2 + \frac{32\pi\rho\omega_k}{k^2 \coth(kh)}|\nu|$, where $|\nu|$ and E were simultaneously determined experimentally. The E_c measurement accuracy was 0.5-2%. The difference between the E_c values obtained by the two methods did not exceed the error of E_c measurement (1-2%).

The surface tension coefficient α was found by means of laser measurements. A traveling wave was created on the water surface by the electromechanical vibrator, and the wave vector k was determined for various wave frequencies ω from the dependence of the wave phase φ on the distance x from the vibrator: φ = kx + const. The distance x was measured with an accuracy of 100 μm. From the data obtained we then determined the parameter α of the dependence (1.1). This method made it possible to obtain the surface tension coefficient with accuracy 2%. However, in replacing the water the spread in α values was 5%. In our series of experiments $\alpha = 57 \pm 3$ erg/cm^2, which is by 22% less than for a surface of pure water.

In preliminary experiments we used distilled water, but it has been observed that after several hours a film of adsorbed particles is formed in it, whose "rigidity" is enhanced with time. Therefore we selected ordinary water, whose "rigidity" is established following several more hours, which made it possible to verify the reproducibility of measurement results obtained in 5-7 days.

The frequency dependence of E_c was measured in the region 4-13 Hz. Figure 3 shows the experimental data, as well as theoretical curves, calculated by Eqs. (1.3), (1.6), (1.12). Curve 1 with $\beta = 160 \pm 20$ erg/cm^2 was obtained by the maximum likelihood method. It differs explicitly from curve 2 with $\beta = \infty$, corresponding to absolute film rigidity.

The accuracy of E_c measurements affected the error in determining the viscosity of water. Experimentally the temperature varied within the limits 16-20°C, therefore the viscosity spread was 4%. This led effectively to a further spread in threshold of order 1%. At the points f = 5, 7, 9 Hz we performed additional measurements. The E_c values were reproduced with an accuracy corresponding to an error 2-2.5%.

From the (f, E_c) data and Eqs. (1.3), (1.6) we find the dependence of wave damping on the wave vector. Figure 4 shows results of experiments and theoretical curves with various β values: curve 5 with β = 0 corresponds to a pure surface, and is described by Eq. (1.10), 1) corresponds to β = 5 erg/cm^2, 2) β = 20 erg/cm^2, 3) with β = 160 erg/cm^2 passes through the experimental points, and 4) with $\beta = \infty$ corresponds to the limit of a rigid lattice and is described by Eq. (1.11). It is noted that the experimentally measured damping γ_k (curve 3) is substantially larger than (1.10) (curve 5), and is in quantitative agreement with the quantity (1.12), obtained in [2], in the whole investigated region from 4 to 13 Hz.

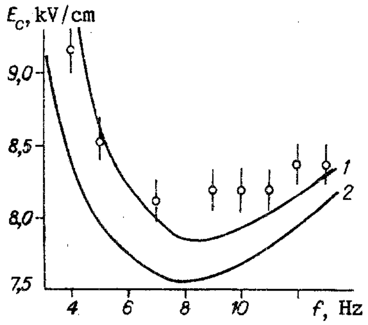

Fig. 3

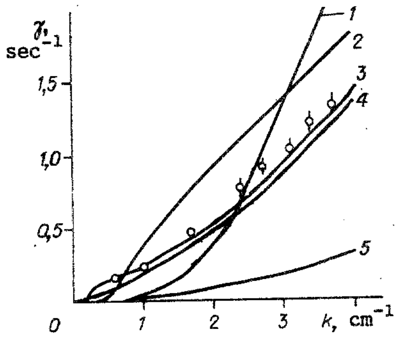

Fig. 4

In processing the experimental results we neglected the dependence of the surface tension coefficient and of other parameters on the electric field, since it was less than 10^{-3} of the intermolecular field.

The following is noted in conclusion: the behavior of waves up to the threshold: $(E_C - E)/E_C \lesssim 0.03$ is well described by the linear theory of parametric instability; the surface tension coefficient of water with particles adsorbed by the surface under experimental conditions is lower than for pure water; the available theory [2] explains the mechanisms and well describes wave damping on a water surface in the frequency region 4-13 Hz; wave damping on the water surface is due to the film surface, and in the frequency region investigated differs by 4-30 times from the value $2\nu k^2$, caused by bulk viscous friction; an essential effect on wave damping occurs due to the dependence of the surface tension coefficient on the adsorbed particle concentration (film "elasticity").

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